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Electric Potentials at Four Points in a Current-Conducting Slab Due to the Presence in the Slab of a Sphere of Differing Conductivity

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A. Introduction

Consider a layer of fluid of conductivity σ, bounded at the surfaces $z = 0$ and $z = t$ by semi-infinite insulating media. Let two parallel bare metal cylindrical electrodes of radius a, where $a \ll t$, lie in the surface $z = 0$, at $x = \pm R$, extending from $y = 0$ to $y = L$. A low-frequency generator feeds the electrodes at the ends $y = 0$. See Fig. 1.

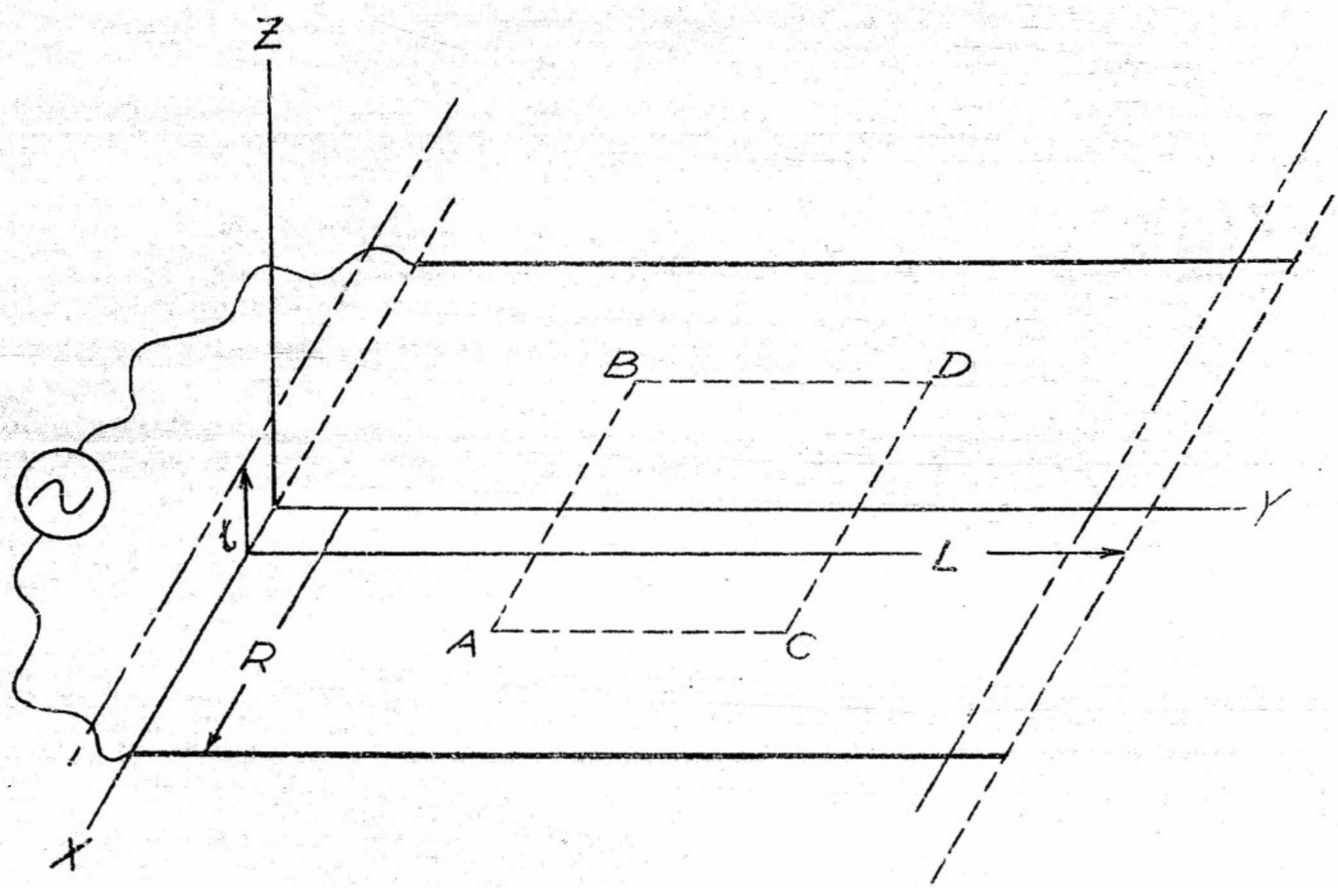

Fig. 1

Now, introduce a relatively small sphere of radius r_o and conductivity σ_o in the region between the electrodes, in the plane $z = 0$. The electrodes will previously have produced a current field, and therefore an electric field, in this region, and the presence of the sphere will produce a distortion in this electric field. It is the problem of this report to calculate, under suitable restrictions, the resulting change of electric potential at the points A, B, C, D, where A, B, C, D form a square array between the electrodes, in the plane $z = 0$.

To arrive at the fields by the complete solution of Maxwell's equations for this special case would be extremely laborious. Instead we shall assume (1) that the frequency of the generator is so low that a direct-current approximation is valid, (2) that the potential along the feeder electrodes varies very slowly with y so that the field in any transverse slice y, y + dy, may be obtained under the assumption that the electrodes are equipotentials, (3) that the electrodes are infinitely long so that end effects are negligible.

With the above assumptions the primary field may be obtained very simply from previous work of D. D. Foster (TR 12)[1] who has worked out the direct-current flow across a long conducting strip due to

equipotential electrodes along opposite sides. The field with the sphere in place is then given, in the d.c. case, for any point outside the sphere, by the sum of the primary field and the field due to a dipole* of a certain strength with its center at the center of the sphere (the sphere imagined removed) and its direction parallel to the direction of the original field. The strength of the dipole is simply related to the strength of the primary, or inducing, field in its neighborhood.

Proceeding in this way, we hope to obtain an order of magnitude calculation for low frequencies.

B. The Calculation

Let $t = 10$ meters

$R = 100$ meters

$L = 300$ meters

$\sigma = 3$ mho/meter

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* Note that, since the medium is conducting, there can be no accumulation of charge in the d.c. case. Thus we have, not an electrostatic problem, but a steady-current problem. The dipole may be regarded either as an electric dipole or as a current dipole, the two being related through the conductivity of the medium. We suggest that thinking in terms of a current dipole tends to less confusion.

Let the locations of the points A, B, C, D be

A:	$x = +50$ m	$y = 100$ m
B:	$x = -50$ m	$y = 100$ m
C:	$x = +50$ m	$y = 200$ m
D:	$x = -50$ m	$y = 200$ m

In TR 12 it was found convenient to assume electrodes of semi-circular cross-section. The electrodes in the actual case to which we wish to approximate are 0000 gauge copper cables with a roughly circular cross-section of outside diameter 1.33 cm. We shall assume for simplicity that the best choice of radius for the semi-cylindrical electrode to give a reasonably equivalent spreading resistance to the actual case is that radius which will give an equal exposed periphery in the conducting medium. Therefore we choose the radius of the semi-cylindrical electrode to be

$$a = 1.33 \times 10^{-2} \text{ meters}$$

Since the electrodes are fed by the generator at one end they may be regarded as a parallel-wire transmission line. The d.c. transmission line equations for the voltage V across the lines and the current I in the lines are

$$\frac{\partial V}{\partial y} = -\mathcal{R} I \qquad (1)$$

$$\frac{\partial I}{\partial y} = -\mathcal{G} V \qquad (2)$$

From handbooks we obtain a value for $\mathcal{R}$, the average resistance per unit length of the transmission line (or of two meters of 0000 gauge copper cable) of

$\mathcal{R} = 3.2 \times 10^{-4}$ ohm/meter

From TR 12, p. 10, we obtain the shunt conductance per unit length of the transmission line

$$\mathcal{G} = \frac{\pi \sigma}{2 \ln\left[\left(\frac{2t}{\pi a}\right) \sinh\left(\frac{\pi R}{t}\right)\right]} \qquad (3)$$

$$= 0.128 \text{ mho/meter}$$

If we neglect any correction for the termination of the line at $y = L$, we obtain for the solution of the transmission line equations

$$V = V_o e^{-(\mathcal{R}\mathcal{G})^{1/2} y} \qquad (4)$$

Substituting the values for $\mathcal{R}$ and $\mathcal{G}$,

$$\frac{V}{V_o} = e^{-0.0064\, y}$$

Values of V/V_o for various values of y are shown in column 2,

Fig. 2. It will be seen that the value of V/V_o at $y = L$ is not entirely negligible, being about 1/7 of the input voltage, and therefore that an appreciable end effect might be expected. However, on account of the tendency of the current flow lines between the outward ends of the electrodes to spread out into the medium, a better approximation to the terminal conditions probably would be to assume some finite terminating conductance rather than to assume an open end, and thus the ignoring of end effects altogether may be as justifiable an approximation to make as any.

TR 12, p. 9, gives, with a simple substitution,

$$E = -\frac{g V_o}{\sigma t}\left[\frac{\sinh\left(\frac{\pi R}{t}\right)}{\cosh\left(\frac{\pi R}{t}\right) - \cosh\left(\frac{\pi x}{t}\right)}\right] \tag{5}$$

for the electric field anywhere in the plane $z = 0$. This formula was arrived at under the assumption that the electrodes were equipotentials. But, proceeding as outlined in part A, we merely substitute V from equation (4) for V_o in equation (5).

A further simplification can be made if we note that, except for points near the lines ($x = \pm R$), the square bracket in (5) is of order unity. Thus, if we restrict

ourselves to $|x| \leqslant$ 75 meters, we have, with negligible error,

$$E = \frac{-\mathcal{J}V_0}{\sigma t} e^{-(R\mathcal{J})^{1/2} y} \tag{6}$$

that is, E is independent of coordinate x, for $|x| \ll$ 75 meters. Substituting numbers,

$$\frac{E}{V_o} = -4.27 \times 10^{-3} e^{-6.4 \times 10^{-3} y} \tag{6a}$$

The values of $\frac{E}{V_o}$ are calculated in column 3 of Fig. 2.

Now it may be shown that the moment of the current dipole due to a perfectly conducting (p.c.) sphere of radius r_o in a uniform conducting medium is given by

$$M_{p.c.} = 4\pi r_o^3 \sigma E \tag{7}$$

and the moment of the current dipole for a perfectly insulating (p.i.) sphere is

$$M_{p.i.} = -2\pi r_o^3 \sigma E \tag{8}$$

The use of equations (7) and (8) is complicated by the proximity of the sphere to the insulating boundary, the plane z = 0. The correct procedure may be made clear by the following argument. Consider a conducting sphere in a uniform field E, the field being in the x-direction. The resulting field configuration will be symmetrical

with respect to a plane through the diameter parallel to the x-direction. The same configuration as far as a point in the upper half space is concerned will be given by the theory of images if one considers a half-sphere in a field E resting on a perfectly reflecting plane. That is to say, where we have a reflecting plane, the correct result for the dipole strength in the region above the reflecting plane will be given by formulas (7) or (8) if we postulate that the object resting on the plane $z = 0$ is a half-sphere of radius r_o. When the object introduced in the actual case is some other shape it seems reasonable that the best approximation should be reached if the half-sphere is chosen to have an equal volume. Thus if we introduce in the physical case a sphere of diameter one meter (radius 0.5 m) we choose r_o in equations (7) and (8) to be

$$r_o = \sqrt[3]{2} \times 0.5$$

Therefore we have, for the physical sphere one meter in diameter

$$\begin{aligned} M_{p.c.} &= 4\pi \times 2(0.5)^3 \sigma E \\ &= \pi \sigma E \end{aligned} \qquad (7a)$$

1.	2.	3.	4.
y	$V/V_o = e^{-0.0064y}$	$E/V_o = .00427\frac{V}{V_o}$	$\frac{M_{p.c.}}{4\pi\sigma V_o}\left(\frac{1}{2t}\right)^2 = 6.25\times10^{-4}\frac{E}{V_o}$
0	1.0	4.27×10^{-3}	2.67×10^{-6}
25	0.852	3.64×10^{-3}	2.27
50	0.726	3.10	1.94
75	0.619	2.64	1.65
100	0.527	2.25	1.41
125	0.449	1.92	1.20
150	0.383	1.63	1.02
175	0.326	1.39	0.87
200	0.278	1.19	0.74
225	0.237	1.01	0.63
250	0.202	0.86	0.54
275	0.172	0.73	0.456
300	0.147	0.63	0.394

Fig. 2

We need consider only the perfectly conducting sphere, since the values for the insulating sphere are simply half in magnitude and opposite in sign.

The potential of a current dipole of moment M, located at (x', y', 0) is, from TR 18[2], p. 4,

$$\phi = \frac{M}{4\pi\sigma} \sum_{n=-\infty}^{n=\infty} \frac{(x-x')}{\left[(x-x')^2+(y-y')^2+(z-2nt)^2\right]^{3/2}} \tag{9}$$

$$= \frac{M}{4\pi\sigma}\left(\frac{1}{2t}\right)^2 \psi$$

where

$$\psi = \sum_{n=-\infty}^{n=+\infty} \frac{\left(\frac{x-x'}{2t}\right)}{\left[\left(\frac{x-x'}{2t}\right)^2+\left(\frac{y-y'}{2t}\right)^2+\left(\frac{z}{2t}-n\right)^2\right]^{3/2}} \tag{10}$$

ψ_0, the function ψ (x, y, z) for x' = y' = 0 has been computed by Mrs. Lois Edelstein by methods developed in TR 18. For other values of x' and y', ψ may be obtained by simple translation. Fig. 4 records the values of ψ at the points A, B, C, D, in the following way. A plan view of the configuration is drawn, divided into 25-meter squares. The center of

each square marks the point at which a dipole has been posited for the computation of its potential field at the points A, B, C, D. The relative values of the potentials, that is, the values of ψ at the four points, are recorded in the four quadrants of each square, as shown in Fig. 3.

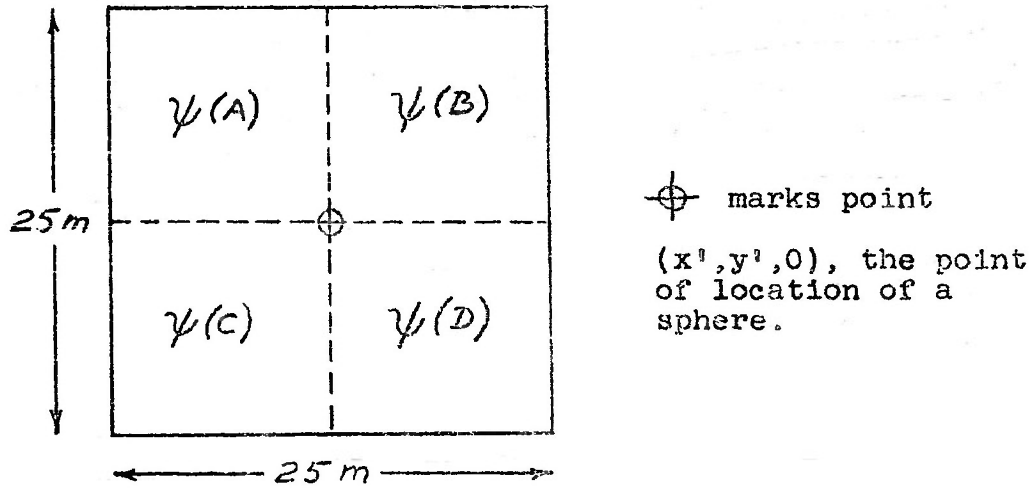

Fig. 3

It remains to multiply each number of Fig. 4 by the appropriate value of $\frac{M}{4\pi\sigma}\left(\frac{1}{2t}\right)^2$ to obtain the values of ϕ, the potential due to a conducting or insulating sphere in the conducting medium. Substituting from (7a), for the perfectly conducting 1 meter sphere,

$$\frac{M_{p.c.}}{4\pi\sigma}\left(\frac{1}{2t}\right)^2 = \frac{\pi\sigma E}{4\pi\sigma}\left(\frac{1}{2t}\right)^2$$

$$= 6.25 \times 10^{-4} E$$

(since t=10 m)

The values of this, in terms of V_o, for various y, are listed in Fig. 2 column 4. The values of ϕ, in terms of V_o, are displayed in Fig. 5, similarly to Fig. 4.

C. Limits of the Results

The potentials at A, B, C, D due to the introduction of a sphere can vary, as noted, from the values in Fig. 5 for a perfectly conducting sphere, to values of half the magnitude and opposite in sign for a perfectly insulating sphere, with, of course, the possibility of zero values for a sphere of average conductivity equal to that of the surrounding medium.

Otherwise, ~~the potentials depend only weakly on the~~ value of σ chosen for the conducting medium, for a constant V_o, since it can be seen that σ cancels out from the equations except where it appears to the 1/2 power in the exponent in equation (4). V_o will not be constant with σ, of course, if a generator of finite impedance is taken into account.

Errors due to the assumption of zero frequency may be expected to become appreciable when (a) the series inductance of the electrodes begins to play an important part in the propagation constant of the transmission line

(the series inductive impedance becomes equal to the series resistance at about 12 c/s), (b) the skin depth of wave propagation in the conducting medium becomes comparable with the dimensions of the system (the frequency for a skin depth of 10 meters is 840 c/s, that for a skin depth of 200 meters is 2.1 c/s). Which of these rough criteria would be the best index to the different behavior at higher frequencies must await a more detailed investigation of the A. C. case.

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References

(1) D. D. Foster, "Flow of Electricity in a Long Flat Strip with Source and Sink on One Edge". ESL Technical Report No. 12 (HPP:590:Serial 6) Edwards Street Laboratory, Yale University, 2 Sept 1952. UNCLASSIFIED

(2) D. D. Foster, "The Potential Due to a Current Dipole in an Infinite Conducting Slab". ESL Technical Report No. 18 (ESL:590:Serial 9) Edwards Street Laboratory, Yale University, 15 May 1953. UNCLASSIFIED

+0.16	+0.23	0.00	+0.26	−0.16	+0.27	−0.25	+0.25	−0.27	+0.16	−0.26	0.00	−0.23	−0.16
+0.032	+0.11	0.00	+0.098	−0.032	+0.082	−0.06	+0.06	−0.082	+0.032	−0.098	0.00	−0.11	−0.032
+0.31	+0.28	0.00	+0.32	−0.31	+0.37	−0.40	+0.40	−0.37	+0.31	−0.32	0.00	−0.28	−0.31
+0.043	+0.13	0.00	+0.12	−0.043	+0.11	−0.08	+0.08	−0.11	+0.043	−0.12	0.00	−0.13	−0.043
+0.80	+0.31	0.00	+0.38	−0.80	+0.48	−0.64	+0.64	−0.48	+0.80	−0.38	0.00	−0.31	−0.80
+0.062	+0.16	0.00	+0.16	−0.062	+0.16	−0.12	+0.12	−0.16	+0.062	−0.16	0.00	−0.16	−0.062
+1.60	+0.32			−1.60	+0.53	−0.80	+0.80	−0.53	+1.60			−0.32	−1.60
+0.095	+0.19	A		−0.095	+0.19	−0.16	+0.16	−0.19	+0.095	B		−0.19	−0.095
+0.80	+0.31	0.00	+0.38	−0.80	+0.48	−0.64	+0.64	−0.48	+0.80	−0.38	0.00	−0.31	−0.80
+0.16	+0.23	0.00	+0.26	−0.16	+0.27	−0.25	+0.25	−0.27	+0.16	−0.26	0.00	−0.23	−0.16
+0.31	+0.28	0.00	+0.32	−0.31	+0.37	−0.40	+0.40	−0.37	+0.31	−0.32	0.00	−0.28	−0.31
+0.31	+0.28	0.00	+0.32	−0.31	+0.37	−0.40	+0.40	−0.37	+0.31	−0.32	0.00	−0.28	−0.31
+0.16	+0.23	0.00	+0.26	−0.16	+0.27	−0.25	+0.25	−0.27	+0.16	−0.26	0.00	−0.23	−0.16
+0.80	+0.31	0.00	+0.38	−0.80	+0.48	−0.64	+0.64	−0.48	+0.80	−0.38	0.00	−0.31	−0.80
+0.095	+0.19			−0.095	+0.19	−0.16	+0.16	−0.19	+0.095			−0.19	−0.095
+1.60	+0.32	C		−1.60	+0.53	−0.80	+0.80	−0.53	+1.60	D		−0.32	−1.60
+0.062	+0.16	0.00	+0.16	−0.062	+0.16	−0.12	+0.12	−0.16	+0.062	−0.16	0.00	−0.16	−0.062
+0.80	+0.31	0.00	+0.38	−0.80	+0.48	−0.64	+0.64	−0.48	+0.80	−0.38	0.00	−0.31	−0.80
+0.043	+0.13	0.00	+0.12	−0.043	+0.11	−0.080	+0.080	−0.11	+0.043	−0.12	0.00	−0.13	−0.043
+0.31	+0.28	0.00	+0.32	−0.31	+0.37	−0.40	+0.40	−0.37	+0.31	−0.32	0.00	−0.28	−0.31
+0.032	+0.11	0.00	+0.098	−0.032	+0.082	−0.06	+0.06	−0.082	+0.032	−0.098	0.00	−0.11	−0.032
+0.16	+0.23	0.00	+0.26	−0.16	+0.27	−0.25	+0.25	−0.27	+0.16	−0.26	0.00	−0.23	−0.16

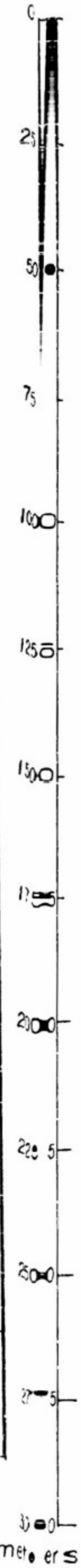

FIG. 4 VALUES of ψ

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+3.6	+5.2	0.00	+5.9	−3.6	+6.1	−5.7	+5.7	−6.1	+3.6	−5.9	0.00	−5.2	−3.6
+0.73	+2.5	0.00	+2.2	−0.73	+1.9	−1.4	+1.4	−1.9	+0.73	−2.2	0.00	−2.5	−0.73
+6.0	+5.4	0.00	+6.2	−6.0	+7.2	−7.8	+7.8	−7.2	+6.0	−6.2	0.00	−5.4	−6.0
+0.83	+2.5	0.00	+2.3	−0.83	+2.1	−1.6	+1.6	−2.1	+0.83	−2.3	0.00	−2.5	−0.83
+13.2	+5.1	0.00	+6.3	−13.2	+7.9	−10.6	+10.6	−7.9	+13.2	−6.3	0.00	−5.1	−13.2
+1.0	+2.6	0.00	+2.6	−1.0	+2.6	−2.0	+2.0	−2.6	+1.0	−2.6	0.00	−2.6	−1.0
+23	+4.5	A		−23	+7.5	−11	+11	−7.5	+23	B		−4.5	−23
+1.3	+2.7			−1.3	+2.7	−2.3	+2.3	−2.7	+1.3			−2.7	−1.3
+9.6	+3.7	0.00	+4.6	−9.6	+5.8	−7.7	+7.7	−5.8	+9.6	−4.6	0.00	−3.7	−9.6
+1.9	+2.8	0.00	+3.1	−1.9	+3.2	−3.0	+3.0	−3.2	+1.9	−3.1	0.00	−2.8	−1.9
+3.2	+2.9	0.00	+3.3	−3.2	+3.8	−4.1	+4.1	−3.8	+3.2	−3.3	0.00	−2.9	−3.2
+3.2	+2.9	0.00	+3.3	−3.2	+3.8	−4.1	+4.1	−3.8	+3.2	−3.3	0.00	−2.9	−3.2
+1.4	+2.0	0.00	+2.3	−1.4	+2.3	−2.2	+2.2	−2.3	+1.4	−2.3	0.00	−2.0	−1.4
+7.0	+2.7	0.00	+3.3	−7.0	+4.2	−5.6	+5.6	−4.2	+7.0	−3.3	0.00	−2.7	−7.0
+0.70	+1.4	C		−0.70	+1.4	−1.2	+1.2	−1.4	+0.70	D		−1.4	−0.70
+12	+2.4			−12	+3.9	−5.9	+5.9	−3.9	+12			−2.4	−12
+0.39	+1.0	0.00	+1.0	−0.39	+1.0	−0.76	+0.76	−1.0	+0.39	−1.0	0.00	−1.0	−0.39
+5.0	+2.0	0.00	+2.4	−5.0	+3.0	−4.0	+4.0	−3.0	+5.0	−2.4	0.00	−2.0	−5.0
+0.23	+0.70	0.00	+0.65	−0.23	+0.60	−0.43	+0.43	−0.60	+0.23	−0.65	0.00	−0.70	−0.23
+1.7	+1.5	0.00	+1.7	−1.7	+2.0	−2.2	+2.2	−2.0	+1.7	−1.7	0.00	−1.5	−1.7
+0.15	+0.50	0.00	+0.45	−0.15	+0.37	−0.27	+0.27	−0.37	+0.15	−0.45	0.00	−0.50	−0.15
+0.73	+1.0	0.00	+1.2	−0.73	+1.2	−1.1	+1.1	−1.2	+0.73	−1.2	0.00	−1.0	−0.73

Scale: 0, 25, 50, 75, 100, 125, 150, 175, 200, 225, 250, 275, 300 meters

ϕ For Values in Volts Multiply by $10^{-7} V_a$

FIG. 5